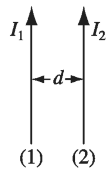
Electricity and Magnetism, Exam 3, 06/04/2018
- with answers -

4 questions, 75 points total

Write your name and student number on the answer sheet. Use of a calculator is allowed. You may make use of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector in the x-direction, and T is a scalar.

1. *(20 points)* Consider two current carrying wires, at a distance d, as depicted in the figure below.



(a) Find the direction and magnitude of the force $\mathbf{F}$ per unit length between the two wires.

Answer:

As an application, let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 (Fig. 5.20). The field at (2) due to (1) is

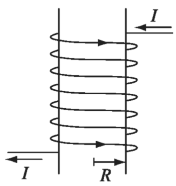
$$B = \frac{\mu_0 I_1}{2\pi d},$$

and it points into the page. The Lorentz force law (in the form appropriate to line currents, Eq. 5.17) predicts a force directed towards (1), of magnitude

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl.$$

The *total* force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}. \tag{5.40}$$



Now consider a very long solenoid, with n windings per unit length, through which a current $\mathbf{I}$ is flowing, as in the figure on the right.

(b) Use Amperian loops to find the magnetic field both inside and outside an infinitely long solenoid.
Answer:

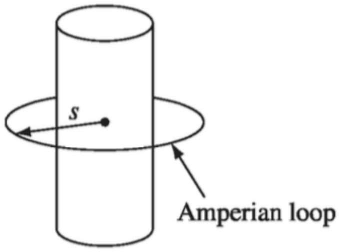


FIGURE 5.36

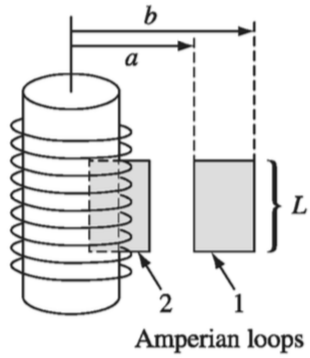


FIGURE 5.37

Fig. 5.37. Loop 1 lies entirely outside the solenoid, with its sides at distances a and b from the axis:

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0,$$

so

$$B(a) = B(b).$$

Evidently the *field outside does not depend on the distance from the axis*. But we agreed that it goes to *zero* for large s. It must therefore be zero *everywhere*! (This astonishing result can also be derived from the Biot-Savart law, of course, but it's much more difficult. See Prob. 5.46.)

As for loop 2, which is half inside and half outside, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L,$$

where B is the field inside the solenoid. (The right side of the loop contributes nothing, since $B = 0$ out there.) *Conclusion:*

$$\mathbf{B} = \begin{cases} \mu_0 n I\, \hat{\mathbf{z}}, & \text{inside the solenoid,} \\ \mathbf{0}, & \text{outside the solenoid.} \end{cases} \tag{5.59}$$

Notice that the field inside is *uniform*—it doesn't depend on the distance from the axis. In this sense the solenoid is to magnetostatics what the parallel-plate capacitor is to electrostatics: a simple device for producing strong uniform fields.

(c) Now consider the top end of the solenoid, where the magnetic fields leave the solenoid. Explain why a piece of diamagnetic material can be made to float above such a solenoid.

Answer: The figure below shows the fringe field of the solenoid, and the force on a current loop - which for this configuration is pointing down. This is why the force is down for paramagnetism. Paramegnetism is magnetization due to the alignment of the magnetic moment of the electrons, associated with their spin. Diamagnetism is a much smaller effect, due to the change in the magnetic moment associated with the orbital motion of the electrons. The sign of the magnetization is different; diamagnetism results in a magnetization antiparallel to the external field. In an inhomogeneous field this results in a force towards the weak-field region. Above a solenoid the magnetic field diverges, and a diamagnetic material is pushed upwards by the magnetic field gradient. If the field is strong enough the *force can counteract gravity, and objects can be made to float.*

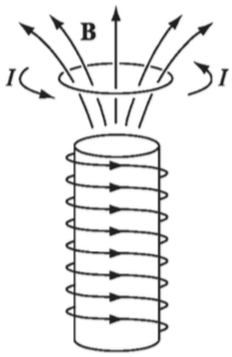


FIGURE 6.3

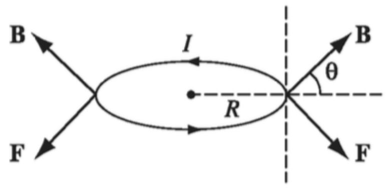
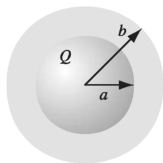


FIGURE 6.4

(d) Suppose an infinitely long solenoid is filled with a paramagnetic material of susceptibility χ_m. Find the magnetic field inside the solenoid, and indicate the direction of the induced surface bound current.

2. *(20 points)*

A metal sphere of radius a carries a charge Q (see figure on the right). It is surrounded, out to a radius b, by linear dielectric material of permittivity ϵ.



(a) Calculate the displacement $\mathbf{D}$ for all points $r > a$.

(b) Calculate the electric field $\mathbf{E}$ for the regions $r < a$, $a < r < b$ and $r > b$.

(c) Calculate the potential at the center (relative to infinity).

(d) Calculate the bound volume charge ρ_b in the dielectric layer.

(e) Calculate the bound surface charge σ_b for the inner and the outer surface of the dielectric layer.

Answer: This problem is given as example 4.5 in the book:

Solution

To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

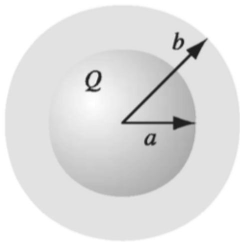


FIGURE 4.20

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right)dr - \int_{a}^{0}(0)\,dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla\cdot\mathbf{P} = 0,$$

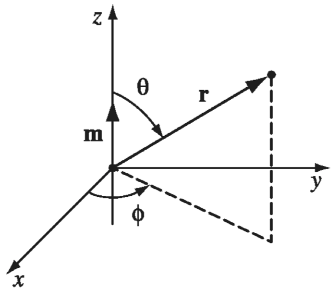
while

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules.

3. *(20 points)* The vector potential for a perfect dipole, located at the origin, pointing in the positive z-direction as in the figure, is given by:

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}.$$

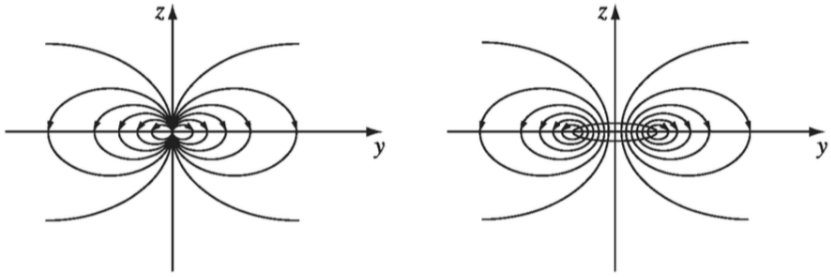


(a) Show that the expression for the magnetic field of this perfect dipole is as follows:

$$\mathbf{B}_{dip}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}).$$

Answer: since $\mathbf{B} = \nabla\times\mathbf{A}$*, you have to take the curl (in spherical coordinates).*

(b) A simple physical dipole can be created by a current loop, with radius R and current I. The current loop is lying flat in the $x-y$ plane, with its center at (0,0,0). Make a sketch of the magnetic field of this physical dipole in the $y-z$ plane. Indicate the region where the field deviates significantly from the formula given in (a).
Answer: The plot of the magnetic field is show below. For comparison also the perfect dipole field is given. The region where the field is significantly different from the perfect dipole expression is in the center of the figure, near the current loop.

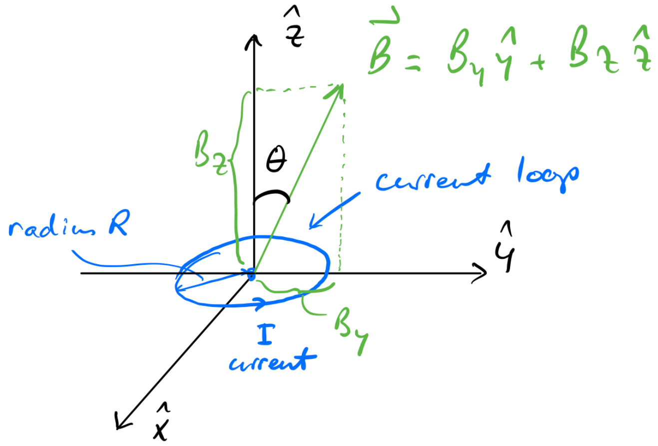


(a) Field of a "pure" dipole (b) Field of a "physical" dipole

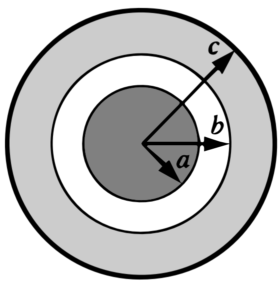
FIGURE 5.55

(c) A homogeneous magnetic field $\mathbf{B} = B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$ is now applied. Using the magnetic dipole moment $\mathbf{m}$ associated with a current loop of this size, calculate the torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ on the current loop.

Answer: For the geometry, see Figure below. The dipole moment of the loop $\mathbf{m} = I\mathbf{a} = I\pi R^2\hat{\mathbf{z}}$, *for a positive current flowing in the loop in the counterclockwise direction when viewed from above. The torque is then* $\mathbf{N} = \mathbf{m}\times\mathbf{B} = I\pi R^2\hat{\mathbf{z}}\times\mathbf{B} = I\pi R^2 B\sin(\theta)\hat{\mathbf{x}}$. *If the angle θ is defined such that it is the angle relative to the magnetic field, then the torque for positive B_y and B_z will be in the $-\hat{\mathbf{x}}$ direction.*



4. *(15 points)* A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (as in the figure below). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown.



(a) Assume that the inner conductor has a charge Q per unit length. Find the electric field $\mathbf{E}$ in the regions $(s < a)$, $(a < s < b)$ and $(b < s < c)$, where s is the radial coordinate.

(b) Find the potential difference between the inner and the outer conductor.

(c) Find the capacitance per unit length of this cable.

Answer: This is problem 4.21 from the book:

Problem 4.21

Let Q be the charge on a length ℓ of the inner conductor.

$$\oint \mathbf{D} \cdot d\mathbf{a} = D 2\pi s \ell = Q \Rightarrow D = \frac{Q}{2\pi s \ell}; \quad E = \frac{Q}{2\pi\epsilon_0 s \ell}\ (a < s < b), \quad E = \frac{Q}{2\pi\epsilon s \ell}\ (b < r < c).$$

$$V = -\int_c^a \mathbf{E} \cdot d\mathbf{l} = \int_a^b \left(\frac{Q}{2\pi\epsilon_0 \ell}\right) \frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\epsilon\ell}\right) \frac{ds}{s} = \frac{Q}{2\pi\epsilon_0 \ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon}\ln\left(\frac{c}{b}\right)\right].$$

$$\frac{C}{\ell} = \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}}.$$

The End